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Randall-Sundrum Choice in the Brane World

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Abstract

We discuss the Randall-Sundrum (RS) choice for h_{MN} in the brane-world. We begin with the de Donder gauge (transverse-tracefree) including scalar(h_{55}), vector($h_{5\mu}$) and tensor($h_{\mu\nu}$) in five dimensions for comparison. One finds that $h_{55} = 0$ and $h_{5\mu} = 0$. This leads to the RS choice. It appears that the RS choice is so restrictive for the five massless states, whereas it is unique for describing the massive states. Furthermore, one can establish the stability of the RS solution with the RS choice only.

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I. INTRODUCTION

Recently there has been much interest in the Randall-Sundrum brane-world [1]. The key idea of this model is that our universe may be a brane embedded in higher dimensional space. A concrete model is a single 3-brane embedded in five-dimensional anti-de Sitter space (AdS_5). Randall and Sundrum have shown that the longitudinal part ($h_{\mu\nu}$) of the metric fluctuations satisfy the Schrödinger-like equation with an attractive delta-function. As a result, the massless zero mode which describes the localized gravity on the brane was found. Furthermore, the massive modes lead to the correction to the Newtonian potential as of $V(r) = G_N \frac{m_1 m_2}{r}(1 + \frac{1}{r^2 k^2})$.

However, we point out that this has been done with the RS choice (a four-dimensional transverse-tracefree gauge). It seems that this choice is so restrictive that the RS model can describe the tensor fluctuation only. Furthermore, in order to have the well-defined theory on the brane, one has to consider the transverse parts of $h_{5\mu}, h_{55}$. More recently, Ivanov and Volovich [2] found that the equation for h_{55} takes the Schrödinger-like equation with a repulsive delta-function. But their linearized equation is not correct.

In the massless and massive cases, h_{55} is a four-dimensional scalar and $h_{5\mu}$ is a four-dimensional vector. Hence it is not natural to set these fields to be zero, as is shown in the RS choice. At the first sight, the RS choice does not seem to be consistent with the massive states. This is because h_{55} and $h_{5\mu}$ belong to the physical fields and these cannot be gauged away because the general covariance is broken. In both cases we choose the other gauge such as the de Donder gauge (a five-dimensional transverse-tracefree gauge) instead of the RS choice in the beginning.

In this paper, we find the correct linearized equation including $h_{5\mu}, h_{55}$. We point out the validity of the RS choice in describing the massless states as well as massive ones in the RS brane model. Also we discuss its connection to the stability of the RS solution.

II. PERTURBATION ANALYSIS

We start with the Einstein equation with the bulk cosmological constant Λ and the brane tension $\tilde{\sigma}$

$$R_{MN} - \frac{1}{2} g_{MN} R = \Lambda g_{MN} + \sigma \sqrt{g_{55}} g_{\mu\nu} \delta^{\mu}_{M} \delta^{\nu}_{N}, \tag{1}$$

which is derived from the action

$$I = \frac{1}{2} \int d^5x \sqrt{g_5} (R + 2\Lambda) + \tilde{\sigma} \int d^4x \sqrt{g_4}. \tag{2}$$

The RS solution is given by

$$\bar{g}_{MN} = H^{-2} \eta_{MN} \tag{3}$$

with $H = k|z| + 1$ and $\eta_{MN} = \mathrm{diag}(+ - - - -)$. Further $\Lambda = -6k^2$ and $\sigma = \tilde{\sigma}\delta(z)$ with $\tilde{\sigma} = 6k$. Here the capital indices $M, N, \cdots$ are split into $\mu, \nu, \cdots$ (four-dimensions: x^{μ}) and $5(x^5 = z)$.

After the conformal transformation of $g_{MN} = \Omega^2 \tilde{g}_{MN}$ with $\Omega = H^{-1}$, let us introduce the perturbation

$$\tilde{g}_{MN} = \eta_{MN} + h_{MN}. \tag{4}$$

Its linearized equation for Eq.(1) takes the form

$$\begin{aligned} &\Box h_{MN} + 3\frac{\partial_K H}{H} \eta^{KL} \left(\partial_N h_{KM} + \partial_M h_{KN} - \partial_K h_{MN} \right) \\ &- \left(\frac{2\Lambda + 2\sigma}{H^2} \right) h_{55} \eta_{MN} - \frac{2\sigma}{H^2} \left\{ h_{MN} - \left(h_{\mu\nu} + \frac{h_{55}}{2} \eta_{\mu\nu} \right) \delta^{\mu}_{M} \delta^{\nu}_{N} \right\} = 0. \end{aligned} \tag{5}$$

Ivanov and Volovich in the version 2 of ref. [2] have missed the second line of Eq.(5). They in the version 3 have missed all of σ-dependent terms but have included the Λ-dependent term. This appears because the terms without ∂ arising from the LHS of Eq.(1) cannot be cancelled against those$\left(H^{-2} \left[\Lambda h_{MN} + \sigma \left(h_{\mu\nu} + \frac{h_{55}}{2} \eta_{\mu\nu} \right) \delta^{\mu}_{M} \delta^{\nu}_{N} \right] \right)$ from the RHS of Eq.(1). This line vanishes if h_{MN} reduces to $h_{\mu\nu}$ with $h_{55} = h_{5\mu} = 0$.

Here we use the de Donder gauge

$$\partial^M h_{MN} = 0, \qquad h^P_P = 0. \tag{6}$$

This means that

$$h^\mu_\mu = h_{55}, \qquad \partial^\mu h_{\mu 5} = \partial_5 h_{55}, \qquad \partial^\mu h_{\mu\nu} = \partial_5 h_{5\nu}. \tag{7}$$

From Eq.(5) we obtain three equations,

$$\left(\Box - \frac{12k^2}{H^2} - 3f\partial_5\right) h_{55} = 0, \tag{8}$$

$$\left(\Box - \frac{12k}{H^2}\delta(z)\right) h_{5\mu} - 3f\partial_\mu h_{55} = 0, \tag{9}$$

$$\left(\Box + 3f\partial_5\right) h_{\mu\nu} - 3f\left(\partial_\mu h_{5\nu} + \partial_\nu h_{5\mu}\right) + \frac{12}{H^2}\left(k^2 - \frac{k}{2}\delta(z)\right) h_{55}\eta_{\mu\nu} = 0 \tag{10}$$

with $f = \partial_5 H/H$. Taking the trace of (10) and comparing it with Eq.(8), one finds that h_{55} should vanish. In deriving $h_{55} = 0$, we use the de Donder gauge in (7). With $h_{55} = 0$, Eq.(9) becomes a decoupled equation. In analyzing the perturbations, if one finds a decouple one, then one should solve it first. And then one has to check its consistency with the remaining equation (10). In order to solve Eq.(9) first, we introduce the separation of variables as

$$h_{5\mu}(x,z) = C_\mu(z)\psi_5(x). \tag{11}$$

Then Eq.(9) takes the form:

$$\left(\Box_4 + m_5^2\right)\psi_5(x) = 0, \tag{12}$$

$$C''_\mu(z) + \left\{\frac{12k}{H^2}\delta(z) + m_5^2\right\} C_\mu(z) = 0 \tag{13}$$

with the gauge condition of $C_\mu(z)\partial^\mu\psi(x) = 0$. Here the prime($'$) means the differentiation with respect to its argument. Now let us solve Eq.(13) first. This is exactly the case of ref. [3]. The solution must satisfy the equation $C''_\mu(z) + m_5^2 C_\mu(z) = 0$ at everywhere, except $z = 0$. And then we assume its plane wave solution as

$$C_\mu(z) = A_\mu e^{-im_5|z|}; \quad C^{z>0}_\mu(z) = A_\mu e^{-im_5 z}, \quad C^{z<0}_\mu(z) = \tilde{A}_\mu e^{im_5 z}. \tag{14}$$

We note that this solution keeps the reflection symmetry of the RS solution as $C_\mu(z') = C_\mu(z)$, under $z' \to -z$. The coefficients in front are the same $A_\mu = \tilde{A}_\mu$ because of the continuity of the wave function. The derivative of $C_\mu(z)$ is no longer continuous because of the presence of the delta-function. That is, one has

$$\frac{\partial C_\mu}{\partial z}\bigg|_{z=0^+} - \frac{\partial C_\mu}{\partial z}\bigg|_{z=0^-} = -12kA_\mu, \tag{15}$$

which leads to

$$im_5 = 6k \tag{16}$$

This admits the tachyonic mass of $C_\mu(z)$ as $m_5^2 = -36k^2 < 0$. In other words, the normalizable bound-state solution to Eq.(13) is allowed if its energy(m_5^2) is negative. As a check, $C^t_\mu(z) = A_\mu e^{-6k|z|}$ satisfies

$$C^{t\,\prime\prime}_\mu(z) + 12k\delta(z)C^t_\mu(z) + m_5^2(= -36k^2)C^t_\mu(z) = 0. \tag{17}$$

But it remains to check whether this solution is or not consistent with Eq.(10). Acting ∂^μ on Eq.(10) and using Eqs.(7) and (9), one gets the condition [4]

$$(\delta(z)C_\mu)' - 3\mathrm{sgn}(z)\delta(z)C_\mu = 0. \tag{18}$$

We note that $\mathrm{sgn}(z)\delta(z)$ is not well defined at $z = 0$ and thus one requires

$$C_\mu(0) = 0. \tag{19}$$

An alternative solution which satisfies Eqs.(13) and (19) is the plane wave as Eq.(14) but $C^p_\mu(0) = 0$,

$$C^p_\mu(z) = A_\mu \sin m_5|z|. \tag{20}$$

At this stage, we remind the reader that our background is AdS_5 with $\delta(z)$-source. This means that the solution to the linearized equations should carry at least the parameter " k" because the size of AdS_5 box is $1/k$ approximately and the brane tension is $\tilde{\sigma} = 6k$. However

this plane wave solution misses "k". This seems to be a solution for 5D Minkowski but not for AdS_5 background. This is so because, due to the condition (19) this does not account for the presence of the brane at $z = 0(12k\delta(z)C_\mu$-term in Eq.(13)) appropriately. On the other hand, if $C_\mu(0) \neq 0$, $\delta(z)C_\mu(z)$ can be taken into account(as in our tachyonic solution C^t_μ). That is, there is no solution which satisfies both Eqs.(9) and (10). Hence we are in a dilemma if $h_{5\mu}$ is truely a massive vector in the RS brane world.

Consequently, the tachyonic solution $C^t_\mu(z)$ is not a physical one because it is incompatible with the tensor equation (10). As it stands, the presence of this solution says that $h_{5\mu}$ should be rejected to have a well-defined theory. Fortunately the consistency with Eq.(10) leads to $h_{5\mu} = 0$ on the whole space z as in the RS choice. Furthermore, the analysis for the massless case ($m_5 = 0$) in Eq.(13) leads to $A_\mu = 0$. This implies that there is no massless vector state on the brane. Hence it is obvious that $h_{5\mu}$ should not be a propagating vector in the RS background. From now on we set $h_{5\mu} = 0$.

III. MASSLESS STATES

These states correspond to $\partial_5 h_{MN} = 0$. Before we proceed, we are willing to count the number of independent components of the graviton h_{MN}. For $D = 5$ dimensions, a symmetric tensor field h_{MN} has $5(5+1)/2 = 15$ independent components, some of which can be eliminated by the gauge conditions (6). This is $-(5+1)$. Further, after choosing the guage (6), there exists a residual gauge degrees of freedom as [5,6]

$$h'_{MN} = h_{MN} - \partial_M \xi_N - \partial_N \xi_M. \tag{21}$$

Notice that h'_{MN} satisfy the de Donder gauge (6) provided that

$$\partial_M \xi^M = 0, \qquad\qquad \Box \xi^M = 0. \tag{22}$$

Thus $(5-1)$ are eliminated by our freedom. Hence the number of massless degrees of freedom in $D = 5$ is

$$\frac{5\cdot 6}{2}-(5+1)-(5-1)=5. \tag{23}$$

In order to see how 5 is composed, let us consider the conventional Kaluza-Klein (KK) model [7,8]. This corresponds to $\Box h_{MN}=0$. Its massless bound state ($\partial_5 h_{MN}=0$) in $D=5$ dimensions can be described by

$$\Box h_{MN}=J_{MN} \tag{24}$$

with the external source J_{MN}. The general covariance of massless case can be represented as a source conservation law of $\partial^N J_{MN}=0$ with $J^M_M=0$. In this case we choose a Lorentz frame as

$$\partial_1=\partial_4, \qquad\qquad \partial_2=\partial_3=\partial_5=0. \tag{25}$$

In this frame, the effective interaction reduces to the positive-definite form

$$\mathcal{L}^{\rm KK}_{\rm massless}=\frac{1}{4}h_{MN}J^{MN}=\frac{1}{4}\sum_{\lambda=-2}^{2}J_{-\lambda}\frac{1}{\partial_1^2-\partial_4^2}J_\lambda, \tag{26}$$

where λ refers to O(2) helicity and

$$J_{\pm 2}=\frac{1}{2}(J_{22}-J_{33})\pm iJ_{23}, \tag{27}$$

$$J_{\pm 1}=J_{52}\pm iJ_{53}, \tag{28}$$

$$J_0=\sqrt{\frac{3}{2}}J_{55}. \tag{29}$$

The terms in (26) with (27), (28), and (29) describe the exchanges of spin-2 graviton, spin-1 photon, and spin-0 scalar. Here we have 2 components (spin-2), 2 (spin-1), and 1 (spin-0) and summing up these leads to 5 in (23). According to the stability analysis [9,10], it is stable if each pole in (26) is positive-definite. Hence the KK model is classically stable.

Now let us consider the same issue using the RS choice as follows:

$$h_{55}=0, \qquad h^\mu_\mu=0, \qquad h_{5\mu}=0, \qquad \partial^\mu h_{\mu\nu}=0. \tag{30}$$

These eliminate -10 in 15. Furthere we point out that there exists a $D=4$ residual gauge as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{31}$$

with

$$\partial_\mu \xi^\mu = 0, \qquad \Box_4 \xi^\mu = 0. \tag{32}$$

This eliminates -3 degrees of freedom (DOFs). Hence we have left 2 DOFs ($= 15 - 10 - 3$) in the RS choice. This is appropriate for describing the graviton $h_{\mu\nu}$ only. Under this gauge, one finds from Eq.(10)

$$\Box_4 h_{\mu\nu} = J_{\mu\nu} \tag{33}$$

with the source relations

$$J_{55} = 0, \qquad J^\mu_\mu = 0, \qquad J_{5\mu} = 0, \qquad \partial^\mu J_{\mu\nu} = 0. \tag{34}$$

Using these, Eq.(26) reduces

$$\mathcal{L}^{\mathrm{RS}}_{\mathrm{massless}} = \frac{1}{4}\Big[J_{-2}\frac{1}{\partial_1^2 - \partial_4^2}J_2 + J_2\frac{1}{\partial_1^2 - \partial_4^2}J_{-2}\Big] \tag{35}$$

with $J_{\pm 2} = J_{22} \pm iJ_{23}$. As a result, the RS choice can describe the massless spin-2 modes of $h_{\pm 2} = h_{22} \pm ih_{23}$ only as it can do best. One cannot find the vector and scalar fields. Three modes of $h_{\pm 1} = h_{52} \pm ih_{53}$ and $h_0 = \sqrt{3/2}h_{55}$ are missed, in comparison with the conventional KK model. Furthermore, it is shown that the RS background is stable because $\mathcal{L}^{\mathrm{RS}}_{\mathrm{massless}}$ in (35) is positive-definite.

IV. MASSIVE STATES

In this case we start with the de Donder gauge in (6). But it turns out that the set of perturbation equations (8)-(10) become

$$(\Box + 3f\partial_5)h_{\mu\nu} = 0, \qquad h_{55} = h_{5\mu} = 0, \tag{36}$$

which corresponds to the RS massive case. At this stage, it is convenient to introduce the new variables $h_{\mu\nu} = H^{3/2}(z)\hat{h}_{\mu\nu}(x,z) = H^{3/2}(z)\psi_h(z)\hat{\hat{h}}_{\mu\nu}(x)$. $\hat{\hat{h}}_{\mu\nu}$ corresponds to the canonical form of $h_{\mu\nu}$ [11]. Then one finds

$$\Big[\frac{\Box_4}{2} + [-\frac{\partial_5^2}{2} + V(z)]\Big]\hat{h}_{\mu\nu} = 0, \tag{37}$$

with

$$V(z) = \frac{15k^2}{8H^2} - \frac{3k}{2H}\delta(z). \tag{38}$$

Considering the equation of $(\Box_4 + m_h^2)\hat{\hat{h}}_{\mu\nu} = J_{\mu\nu}$ with $J_{55} = J_{5\mu} = 0$, we find the source conservation law as follows:

$$\partial^\mu J_{\mu\nu} = 0 \qquad\qquad J^\mu_\mu = 0. \tag{39}$$

Further the mass m_h^2 is determined by the equation [1]

$$\Big[-\frac{1}{2}\partial_5^2 + V(z)\Big]\psi_h(z) = \frac{1}{2}m_h^2\psi_h(z). \tag{40}$$

It was shown that $V(r)$ guarentees $m_h^2 \geq 0$ [11]. This implies that these are no normalizable negative energy graviton modes. In this case we choose a massive Lorentz frame in which

$$\partial_1 = \partial_5 \qquad \text{and} \qquad \partial_i = 0 \qquad \text{for} \qquad i = 2, 3, 4. \tag{41}$$

It follows that, in the neighborhood of the pole, the effective interaction reduces to

$$\mathcal{L}^{\rm RS}_{\rm massive} = \frac{1}{4}J_{ij}\frac{1}{\partial_1^2 + m_h^2}J_{ij}, \tag{42}$$

where J_{ij} is a symmetric traceless tensor in three dimensions [7].

One finds a massive tensor with 5 DOFs because J_{ij} has 5 $(= 3 \cdot 4/2 - 1)$ components. It is interesting to ask how we can interpret this DOFs. This is clear from the fact that in the massive case the global symmetry of spacetime is spontaneously broken [8]. The gauge parameters $\xi_\mu(x, z)$ and $\xi_5(x, z)$ in (21) are associated to spontaneously broken generators. The $h_{\mu\nu}$ with 2 DOFs acquire mass m_h^2 by eating 2 DOFs of $h_{5\mu}$-vector and 1 DOF of h_{55} scalar. Thus one finds a pure spin-2 massive particle with 5 DOFs. Explicitly, these are h_{23}, h_{24}, h_{34}, and other two satisfying $h_{22} + h_{33} + h_{44} = 0$ [9]. All these have positive-definite norm states. Hence all of the massive states in the RS model are classically stable.

V. DISCUSSION

We study the validity of the RS choice in the RS model. For this purpose we start with the de Donder gauge. Using the RS choice for the massive case, one finds 5 DOFs in $h_{\mu\nu}$. These all turn out to be the physical massive modes. Hence there remains no residual gauge symmetry. In the massless case, we have three gauge degrees of freedom upon choosing the RS one. This corresponds to a residual gauge degrees of freedom. Using these, we can always find the massless spin-2 with 2 DOFs; for example, see Ref. [6]. Hence we always have a localized gravity in a 3-brane.

For the stability of the RS solution of $ds^2_{\rm RS} = H^{-2}\eta_{MN}dx^M dx^N$, we find that $\mathcal{L}^{\rm RS}_{\rm massless}$ has positive norm states for the RS choice, and also $\mathcal{L}^{\rm RS}_{\rm massive}$ is positive-definite with the RS choice. This means that the RS choice is regarded as a unique one to establish the stability of the RS solution.

Finally, we comment on the Schrödinger-like equation (9) with $h_{55} = 0$. It may imply that the RS model is not classically stable because it has a tachyonic mass. However, considering its consistency with Eq.(10), this puzzle can be resolved. If the RS model makes sense, $h_{5\mu} = 0$.

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